Lecture 15 :Maxima and Minima

In this section we will study problems where we wish to find the maximum or minimum of a function. For example, we may wish to minimize the cost of production or the volume of our shipping containers if we own a company. There are two types of maxima and minima of interest to us, Absolute maxima and minima and Local maxima and minima.

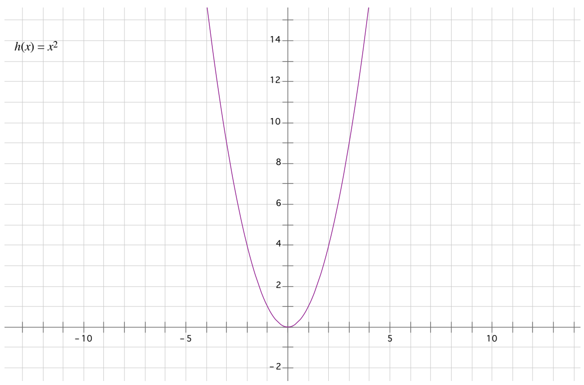
Absolute Maxima and Minima

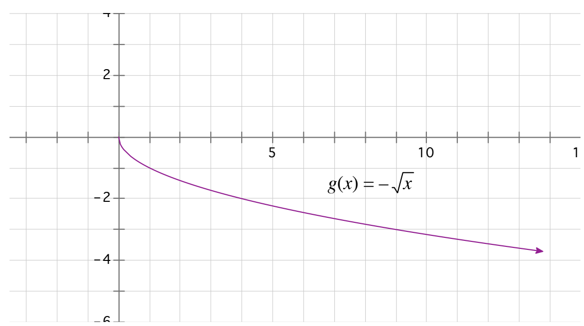
Definition f has an **absolute maximum** or global maximum at c if $f(c) \geq f(x)$ for all x in $D =$ domain of f. $f(c)$ is called the maximum value of f on D.

Definition f has an **absolute minimum** or global minimum at c if $f(c) \leq f(x)$ for all x in $D =$ domain of f. $f(c)$ is called the minimum value of f on D.

Maximum and minimum values of f on D are called extreme values of f.

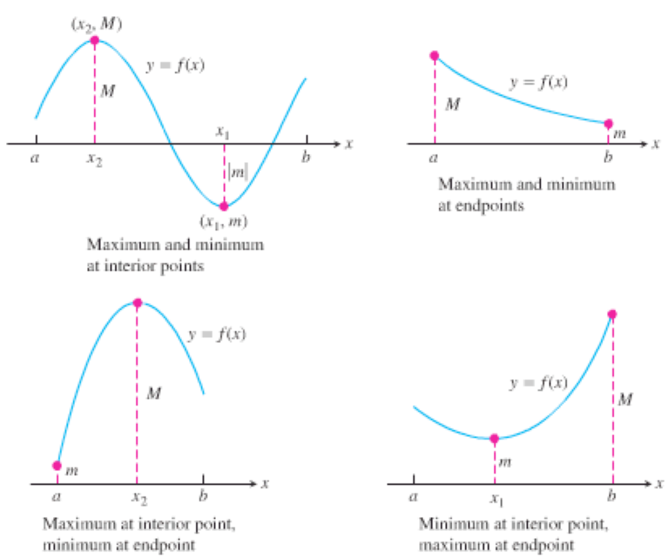
Example Consider the graphs of the functions shown below. What are the extreme values of the functions; $h(x) = x^2$ and $g(x) = -\sqrt{x}$?





Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers c and d in $[a, b]$ with $f(c) = M$ and $f(d) = m$ and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

This can happen in a variety of ways. We can see some of the possibilities in the picture below.



Example If $f(x) = \sin x$, what is the absolute maximum and absolute minimum of $f(x)$ on the interval $0 \leq x \leq 2\pi$?

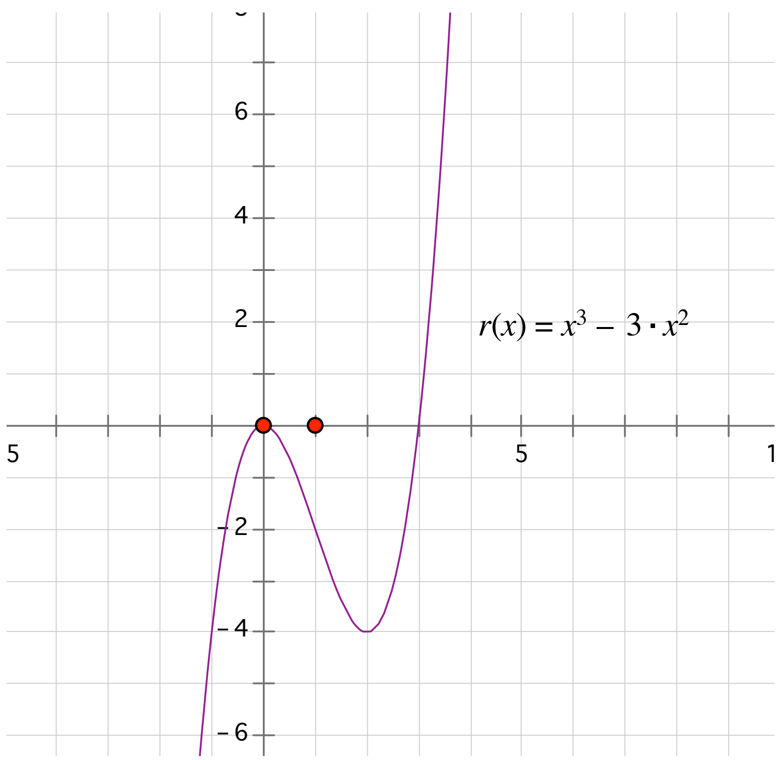
Note This theorem does not apply to functions which are not continuous on $[a, b]$.

Example $f(x) = 1/x$ on the interval $[-1, 1]$. Draw a graph to see what happens.

We see that some graphs have points that are maxima or minima in their neighborhood, but are not absolute maxima or minima.

Definition A function f has a **local maximum** at a point c if $f(c) \geq f(x)$ for all x in some open interval containing c. A function f has a **local minimum** at a point c if $f(c) \leq f(x)$ for all x in some open interval containing c.

Example The graph of $r(x) = x^3 - 3x^2$ is shown below. Find the points where the function has local maxima and minima.



We us the following theorem to identify potential local maxima and minima.

Theorem (Fermat's Theorem) If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Proof Suppose f has a local maximum at c. Then $f(c) \geq f(x)$ when x is near c. The derivative of f at c must equal the following right hand limit

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}.$$

Since $f(c+h) \leq f(c)$ when h is small and $h > 0$ in the above limit, we have that $\frac{f(c+h)-f(c)}{h} \leq 0$, hence

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq \lim_{h \to 0^+} 0 = 0.$$

This gives us that $f'(c) \leq 0$. On the other hand $f'(c)$ must also equal the left hand limit:

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}.$$

Here $h < 0$ and $f(c+h) - f(c) \leq 0$ hence we have that $\frac{f(c+h)-f(c)}{h} \geq 0$ and

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq \lim_{h \to 0^-} 0 = 0.$$

This gives us that $f'(c) \geq 0$. The only number that can be ≥ 0 and ≤ 0 is 0 itself. Hence

$$f'(c) = 0.$$

The proof for a local minimum is similar.

Example Consider the function $r(x) = x^3 - 3x^2$ shown above. Verify that $r'(0)$ and $r'(2)$ are equal to zero.

We must keep in mind the following points when using this theorem:

- If a function has a point c where $f'(c) = 0$, it does NOT imply that the function has a local maximum or minimum at c.

  Example $f(x) = x^3$ at $x = 0$

- A function may have a local maximum or minimum at a point where the derivative does not exist.

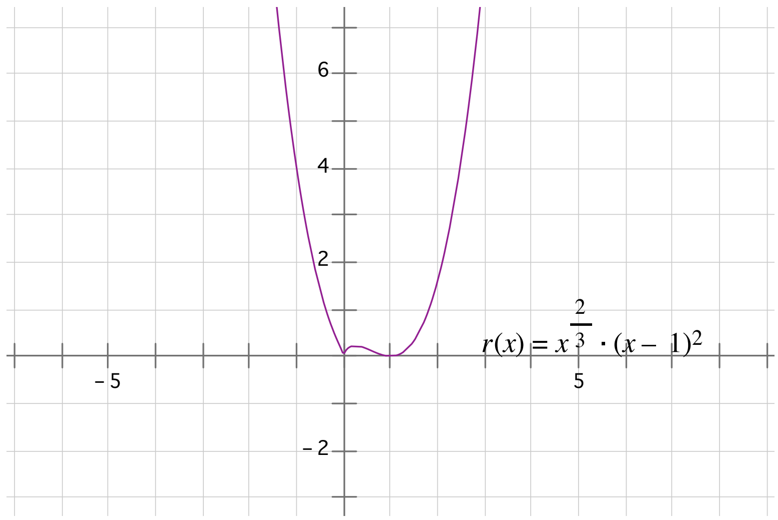
  Example $g(x) = |x|$ at $x = 0$.

Nevertheless identifying the points where $f'(c) = 0$ helps us to find local maxima and minima.

Critical Points/Critical Numbers

Definition A **critical number/point** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example Find the critical numbers of the function $r(x) = x^{2/3}(x-1)^2$.



.

Note By Fermat's theorem above, if f has a local maximum or minimum at c, then c is a critical number of f.

Finding the absolute maximum and minimum of a continuous function on a closed interval $[a, b]$.

To find the **absolute** maximum and minimum values of a continuous function f on a closed interval $[a, b]$;

1. Find all of the critical points of f in the interval $[a, b]$.
2. Evaluate f at all of the critical numbers in the interval $[a, b]$.
3. Evaluate f at the endpoints of the interval, (calculate $f(a)$ and $f(b)$.)
4. The largest of the values from steps 2 and 3 is the absolute maximum of the function on the interval $[a, b]$ and the smallest of the values from steps 2 and 3 is the absolute maximum of the function on the interval $[a, b]$.

Example Find the absolute maximum and minimum of the function $r(x) = x^{2/3}(x-1)^2$ on the interval $[-1, 1]$.

Note Sometimes the absolute maximum can occur at more than one point c. The same is true for the absolute minimum.

Example Find the absolute maximum and minimum of the function $f(x) = x^3 - 3x^2$ for $1 \leq x \leq 4$.

Example The profit function for my company depends (partly) on the number of widgets I produce. The relationship between $x =$ the number of widgets I produce and my profits (all other variables remaining constant) is given by

$$P(x) = 4 + 0.03x^2 - 0.001x^3.$$

Find the production level for widgets that will maximize this function if I have the capacity to produce at most 50 widgets.

Since production is limited to $0 \leq x \leq 50$, we must maximize the profit function $P(x) = 4 + 0.03x^2 - 0.001x^3$ on the interval $[0, 50]$. $P(x)$ is continuous on this interval since it is a polynomial, therefore by the Extreme value theorem $P(x)$ has an absolute maximum on the interval. Following our 3 step proceedure:

1. **Critical Points** $P'(x) = 0.06x - 0.003x^2$. All values of x in the interval $[0, 50]$ are in the domain of P and in the domain of P', so the critical points occur where $P'(x) = 0$.

$$P'(x) = 0.06x - 0.003x^2 = 0.003x(20 - x) = 0$$

if $x = 0$ or $x = 20$.

$$\boxed{\text{Critical points} \quad x = 0 \text{ and } \quad x = 20}$$

2. **Evaluate at critical points** $P(0) = 4, \quad P(20) = 4 + 0.03(20) - 0.003(20^2) = 8.$

3. **Evaluate at end points** $P(0) = 4, \quad P(50) = 4 + 0.03(50) - 0.003(50^2) = -46.$

4. **Choose the largest value** Absolute maximum at $x = 20$. $P(20 = 8$ is the absolute maximum profit in this production range.